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ACTIONS OF DIHEDRAL GROUPS ON POLYNOMIALS

О. Овруцкая Действие группы отражений может быть обобщено на тензорное и диагональное действие степени группы отражений на многомерные полиномы. В результате получаем алгебру диагональных инвариантов, которая является свободным модулем относительно алгебры тензорных инвариантов. В данной работе мы рассматри-ваем действия диэдральной группы $I_2(m)$. Полиному, который является отношением двух Гильбертовых рядов, мы даём новое представление в терминах “flag major index" для любой тензорной степени. Данная работа дополняет подобные исследования для групп Коксетера типов *A, B и D.*

O.Ovrutzki. The natural action of a reflection group can be extended to a tensor action of its power on multivariate polynomials, as well as to a diagonal action of the group itself on the same polynomials. The resulting algebra of diagonal invariants is a free module over the algebra of tensor invariants. In this work we consider the case of the dihedral groups $I_2(m)$. We determine explicitly the polynomial, which is the quotient of the two Hilbert series, and express it in terms of a new “flag major index" statistic on the group, for any tensor power k. This extends previous work for the Coxeter groups of types *A, B* and *D*.

Introduction.

1.1. Outline.

The notion of major index (see below) in the symmetric group was introduced by MacMahon about a hundred years ago.

Let Sn be the symmetric group on n letters. Sn is a Coxeter group with respect to the Coxeter generating set $S=\{s_i \mid 1\le i<n\}$ where $s_i:=(i,i+1),\quad (1\le i<n)$ are the adjacent transpositions.

Let the length of a permutation π be the minimal number of factors in an expression of π as a product of the

Coxeter generators $(i,i+1),\quad (1\le i<n)$.

Let $Des(\pi):=\{1\le i<n \mid \ell(\pi s_i)<\ell(\pi)\}$ be the descent set of π, and let

$$maj(\pi):=\sum_{i\in Des(\pi)} i$$

Denote the length of $\pi \in S_n$ by $\ell(\pi)$.

be the major index of π.

MacMahon gave a simple product formula for the generating function of major index over the symmetric group; the major index was shown to be equidistributed with the length function, [11].

The search for an extended major index and corresponding identities on other groups, initiated by Foata in the early nineties, turned out to be successful for the classical Weyl groups and some wreath products. In particular, the Hilbert series of the coinvariant algebra of the symmetric group *Sn* and of the wreath products $S_n \sim \square_r$ may be expressed as generating functions for the flag major index on these groups.

It follows from Theorem 2.4 *that the quotient of the Hilbert series for the diagonal*

and tensor invariant algebras, respectively, is a polynomial.

The combinatorial proofs of this fact for the classical Weyl groups was extended to *Sn* by Garsia and Gessel [6]; to *Bn* by Adin, Brenty and Roichman [1]; to *Dn* by Biagioli and Caselli [4]; to wreath products $S_n \sim \square_r$ by Adin and Roichman, [2]; to complex reflection groups by Shwartz, Adin and Roichman[3]. All these groups have “permutation-like" elements. Consequently, the following problem was posed: extend the notion of (flag) major index to other groups, e.g., to (non-classical) Coxeter groups. The current work seems to be the first result in this direction. Here we extend the notion of flag major index to the Coxeter groups of type $I_2(m)$, namely the dihedral groups (Theorem2.10). We find a corresponding basis for the diagonal invariant algebra (Theorem 3.2) and for the coinvariant algebra (Lema3.3). A corresponding basis for the diagonal invariant algebra for *Bn* can be found in [2].

2. Background

2.1. Group Actions on Multivariate Polynomials.

Let *V* be an *n*-dimensional vector space over a field *F* of characteristic 0. Let *G* be a finite subgroup of *GL(V)*. It acts in a natural way on the ring of polynomial functions on *V*, which may be identified with the polynomial ring *Pn* = F [$x_1, x_2, \ldots, x_n$], where x_i are the coordinate functions.

Denote the natural action of *G* on *Pn* by φ. The tensor power $P_n^{\otimes k} := P_n \times \ldots \times P_n$ (*k*-times) may be interpreted as the ring of polynomials in *k* sets of *n* variables each. It admits a natural *tensor action*

φ_T of $G_k = G \times \ldots \times G$ (k-times):

$$\varphi_T(g_1,\ldots g_k)(f_1 \otimes \ldots \otimes f_k) := \varphi(g_1)(f_1) \otimes \ldots \otimes \varphi(g_k)(f_k)$$

The diagonal embedding $d : G \to G^k$ defined by $g \mapsto (g_1,...g_k) \in G^k$ $(\forall g \in G)$ gives rise to a *diagonal action* $\varphi_D = \varphi_T \circ d$ of G on $P_n^{\otimes k}$:

$$\varphi_D(g)(f_1 \otimes ... \otimes f_k) := \varphi(g)(f_1) \otimes ... \otimes \varphi(g)(f_k).$$

The corresponding invariants form *the tensor invariant algebra* (TIA)

$$TIA := \left\{ \overline{p} \in P_n^{\otimes k} \;\; | \varphi_T(\overline{g})(\overline{p}) = \overline{p}, \quad \forall \overline{g} \in G^k \right\}$$

and the *diagonal invariant algebra* (DIA)

$$DIA := \left\{ \overline{p} \in P_n^{\otimes k} \;\; | \varphi_D(\overline{g})(\overline{p}) = \overline{p}, \quad \forall \overline{g} \in G \right\}$$

These two algebras are naturally multigraded (by degree in each copy of Pn).

Consider the corresponding Hilbert series

(1) $F_D(\overline{t}) := \sum_{i_1,...,i_k \in \mathbb{N}} \left(\dim_F DIA_{i_1,...,i_k} \right) t_1^{i_1} \cdots t_k^{i_k}$

and

(2) $F_T(\overline{t}) := \sum_{i_1,...,i_k \in \mathbb{N}} \left(\dim_F TIA_{i_1,...,i_k} \right) t_1^{i_1} \cdots t_k^{i_k}$

where $\overline{t} = (t_1,...,t_k)$, $\mathbb{N}$ is the set of nonnegative integers, and $DIA_{i_1,...,i_k} \left(TIA_{i_1,...,i_k} \right)$

is the homogeneous piece of multi-degree $(i_1,...,i_k)$ in *DIA* (*TIA,* respectively), [2].

Theorem 2.1. *(Molien, 1897), [7] If G is a finite group with a linear representation*

$\rho :\; G \to GL(V)$, *then*

(3) $H\left(S(V)^G, t\right) = \frac{1}{|G|} \sum_{g \in G} \det\left(1 - \rho(g)t\right)^{-1}.$

By Molien's Theorem 2.1, applied to φ_T *and* φ_D*,*

(4) $F_T(\overline{t}) := \prod_{i=1}^{k} \frac{1}{|G|} \sum_{g \in G} \det\left(1 - \rho(g)t_i\right)^{-1}$

and

(5) $F_D(\bar{t}) := \frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{k}\det(1-\rho(g)t_i)^{-1}$

In certain important cases, the quotient of these two multivariate Hilbert series is a polynomial with nonnegative integer coefficients. This happens, for instance, whenever the algebra DIA is Cohen-Macaulay (e.g., whenever we are working over a field of characteristic zero, or finite characteristic not dividing $|G|$). We list some fundamental results, which will be used in our proofs.

Theorem 2.2. *(Hochster and Eagon) [8] Suppose that G is a complex reflection group and $|G|$ is invertible in the field K. Then $S = K[t_1, t_2, ..., t_n]$, is a finitely generated free R-module.*

Theorem 2.3. *(Shephard-Todd, Chevalley, Serre) [9] The following statements*

on a finite group G are equivalent:

(1) G is a complex reflection group.

(2) $S(V)$ is a free finitely generated $S(V)^G$-module.

(3) $S(V)^G$ is a polynomial algebra.

2.2. Actions of reflection groups. *The following theorem proves that DIA, the diagonal invariant algebra is a free module over its (polynomial) subalgebra -TIA , the tensor invariant algebra.*

Assume now that G $\subseteq$ GL(V) is a finite reflection group.

Theorem 2.4. *Let G $\subseteq$ GL(V) be a finite reflection group acting on $P_n^{\otimes k}$ via the diagonal action. Then DIA is a free module of rank $|G|^{k-1}$ over TIA.*

Proof: Let $H = G^k$ be the cartesian power of G.

H acts on $\bigoplus_{i=1}^{n} V_i \cong F^{nk}$ and thus on $P_n^{\otimes k} \cong P_{nk}$ as a reflection group, and therefore by Theorem 2.3 its invariant algebra TIA = P_{nk}^H is a polynomial ring: $TIA = F[y_1', ..., y_{nk}']$. G acts on P_{nk} via the diagonal action. The invariant algebra DIA = P_{nk}^G is Cohen-Macaula[1]. Thus there exists a homogeneous system of parameters $\{y_1, ..., y_{nk}\}$ such that DIA is a finitely generated $F[y_1, ..., y_{nk}]$-module. We explain later why the number of parameters is equal to nk.

By Theorem 2.7, the algebra DIA is a free module over $F\left[y_1',\ldots,y_{nk}'\right]$ *for any homogeneous system of parameters* $y_1',\ldots,y_{nk}'$. *TIA* $\subseteq$ *DIA is a polynomial ring.*

Since $0\neq|G|<\infty$ *and* $R = P_{nk}$ *is a finite by generated* R^G*-module, by [27, Remark 5.20],*
R is integral over R^G. *Since R is integral over* R^G,

$K\dim R^G = K\dim R;$

by [27, Cor. 6.33], where K dim *denotes the Krull dimension, see the definition in [16].*

Similar arguments for R^{G^k} *yield* $K\dim R^{G^k} = K\dim R.$

Thus $K\dim R^{G^k} = K\dim R^G$ *Thus DIA and TIA have the same Krull dimension equal to nk, so the generators* $y_1',\ldots,y_{nk}'$ *of TIA form a homogeneous system of parameters for DIA.*

Conclusion: DIA is a free module over its (polynomial) subalgebra TIA. Let us calculate the rank of DIA over TIA.

An old result of Hochster and Eagon(see Theorem 2.2 above) shows that the polynomial ring is a free module over the invariant ring, in our case TIA. However, H is a reflection group. DIA is a free module over TIA, thus from the proof of the Theorem 2.2 [8] it follows that the rank of P_{nk} *over* P_{nk}^G *is equal to* $|G|$, *and the rank of* P_{nk} *over* $P_{nk}^{G^k}$ *is equal to* $|G|^k$.

The rank of DIA over TIA is thus equal to $|G|^{k-1}$.

Q.E.D

Corollary 2.5. *Let* $G\subseteq GL(V)$ *be a finite reflection group acting on* $P_n^{\otimes k}$ *via the diagonal action. Then the quotient* $\frac{F_D(\bar{t})}{F_T(\bar{t})}$ *is a polynomial with* $|G|^{k-1}$ *summands.*

This result was given a combinatorial proof with an explicit basis demonstratingit in several important cases: There are combinatorial proofs of this fact for the classical Weyl groups;

for Sn by A. Garsia and I. Gessel [6]; for Bn by R. Adin and Y. Roichman [2]; for Dn by R. Biagioli and F. Caselli [4]. We shall prove it, with an explicit formula for the quotient, for the dihedral groups I2(m). We shall prefer the following presentation of the dihedral groups

(6) $$I_2(m)=\left\langle r,f \;\middle|\; r^m=f^2=1,\ frf^{-1}=r^{-1}\right\rangle$$

where r, f may be interpreted as a rotation by $2\pi/m$ *and a reflection of the m-gon, respectively. We can take for instance: r = st, f = t.*

The corresponding sets of 2×2 *matrices are the (subgroup of) "rotation" matrices.*

(7) $$G_{rot} := \left\{\rho\left(r^{\alpha}\right) : 0 \le \alpha \le m-1\right\} = \left\{\begin{pmatrix} e^{2\pi i\alpha/m} & 0 \\ 0 & e^{-2\pi i\alpha/m} \end{pmatrix} : 0 \le \alpha \le m-1\right\}$$

and the (coset of) "reflection"matrices

(8) $$G_{ref} := \left\{\rho\left(r^{\alpha} f\right) : 0 \le \alpha \le m-1\right\} = \left\{\begin{pmatrix} 0 & e^{2\pi i\alpha/m} \\ e^{-2\pi i\alpha/m} & 0 \end{pmatrix} : 0 \le \alpha \le m-1\right\}$$

Definition 2.6. [3], [20] *Let G be a finite group. A sequence* $(r_1,...,r_k)$ *of elements of G is called an ordered generating system (or ordered basis) for G ifevery element* $g \in G$ *has a unique representation* $g = r_1^{\alpha_1} \cdots r_k^{\alpha_k}$, *where* $0 \le \alpha_i < |r_i|$ *for every 1≤ i≤ k (*$|r_i|$ *is the order of* r_i *).*

Thus, the system of generators (r; f) for $I_2(m)$ *is an ordered generating system.*

2.3. Hilbert series for the dihedral group.

Following previous work on other groups [2] [4] , we introduce the following

Definition 2.7. *The flag major index of an element* $g = r^{\alpha} f^{\beta}$ $(0 \le \alpha \le m,\quad 0 \le \beta \le 2)$
of the dihedral group is

(9) $$fmaj(g) := \alpha + \beta .$$

Using $fr = r^{-1} f$, *we can write the product rules in* $I_2(m)$

(10) $$r^{\alpha_1} \cdot r^{\alpha_2} = r^{\alpha_1+\alpha_2}, \qquad r^{\alpha_1} \cdot r^{\alpha_2} f = r^{\alpha_1+\alpha_2} f$$
$$r^{\alpha_1} f \cdot r^{\alpha_2} = r^{\alpha_1-\alpha_2} f, \qquad r^{\alpha_1} f \cdot r^{\alpha_2} f = r^{\alpha_1-\alpha_2}.$$

In short,

(11) $$r^{\alpha_1} f^{\beta_1} \cdot r^{\alpha_2} f^{\beta_2} = r^{\alpha_1+(-1)^{\beta_1}\alpha_2} f^{\beta_1+\beta_2}.$$

Let $I_2(m)^k$ *be a cartesian power of the dihedral group*

$$I_2(m)^k = G_1 \times G_2 \times \cdots \times G_k,\quad G_i \cong I_2(m)\quad (\forall i).$$

Its elements have the form $\left(r_1^{\alpha_1} f_1^{\beta_1}, r_2^{\alpha_2} f_2^{\beta_2}, ..., r_k^{\alpha_k} f_k^{\beta_k}\right)$ *where* $r_i\ f_i \in G_i,\quad 0 \le \alpha_i < m,\quad 0 \le \beta_i < 2 \quad (1 \le i \le k)$.

Theorem 2.8. *Let* $G = I_2(m)$ *act naturally on* $P_2^{\otimes k}$.

Denote $H := \left\{ \bar{i} = (i_1, ..., i_k) : \ -m < i_1, ..., i_k < m, \quad \sum_{s=1}^{k} i_s \equiv 0 (\operatorname{mod} m) \right\}$.

Then $\dfrac{F_D(\bar{t})}{F_T(\bar{t})} = \dfrac{1}{2}\left(\sum_{i \in H} t_1^{|i_1|} \cdots t_k^{|i_k|} + \prod_{s=1}^{k} (1 - t_s^m) \right)$.

Proof: By Theorem 2.4, the quotient $\dfrac{F_D(\bar{t})}{F_T(\bar{t})}$ *of the Hilbert series for the diagonal and tensor invariant algebras, respectively, is a polynomial. We want to compute it explicitly. Clearly,*

(12) $$F_T(\bar{t}) = \prod_{s=1}^{k} \left[\frac{1}{2m} \sum_{g \in I_2(m)} \frac{1}{\det(1 - \rho(g) t_s)} \right] = \prod_{s=1}^{k} \left[\frac{1}{(1 - t_s^m)(1 - t_s^2)} \right]$$

and

$$F_D(\bar{t}) = \left[\frac{1}{2m} \sum_{g \in I_2(m)} \frac{1}{\prod_{s=1}^{k} \det(1 - \rho(g) t_s)} \right]$$

Calculate the "rotation" and the "reflection" parts of this sum separately.

$$F_D(\bar{t}) = \frac{1}{2m} \left[\sum_{g \in G_{ref}} \frac{1}{\prod_{s=1}^{k} \det(1 - \rho(g) t_s)} + \sum_{g \in G_{rot}} \frac{1}{\prod_{s=1}^{k} \det(1 - \rho(g) t_s)} \right]$$

where G_{ref} *and* G_{rot} *are subsets of* $I_2(m)$ *as defined in (8) and (7), respectively.*

Clearly,

(13) $$\sum_{g \in G_{ref}} \frac{1}{\prod_{s=1}^{k} \det(1 - \rho(g) t_s)} = \frac{m}{\prod_{s=1}^{k} (1 - t_s^2)}$$

while

(14) $$\sum_{g \in G_{rot}} \frac{1}{\prod_{s=1}^{k} \det(1 - \rho(g) t_s)} = \frac{p(\lambda^j t_s)\, p(\lambda^{-j} t_s)}{\prod_{s=1}^{k} (1 - t_s^m)^2},$$

where $p(t) = \sum_{i=0}^{m-1} t^i$.

The proof requires the following lemma.

Lemma 2.9. $$p\left(\lambda^j t\right) p\left(\lambda^{-j} t\right) = \frac{\left(1-t^m\right)\left(\sum_{n=-(m-1)}^{m} \lambda^{jn} t^{|n|}\right)}{1-t^2}.$$

Proof:
(15)
$$p\left(\lambda^j t\right) p\left(\lambda^{-j} t\right) = \sum_{i=0}^{m-1} \lambda^{ij} t^i \cdot \sum_{i=0}^{m-1} \lambda^{-ij} t^i = 1 + \left(\lambda^j + \lambda^{-j}\right) t + \left(\lambda^{2j} + 1 + \lambda^{-2j}\right) t^2 + \ldots + t^{2m-2}$$
is a polynomial in t of degree $2m-2$.
Thus
$$\left(1-t^2\right) p\left(\lambda^j t\right) p\left(\lambda^{-j} t\right) = 1 + \left(\lambda^j + \lambda^{-j}\right) t + \left(\lambda^{2j} + \lambda^{-2j}\right) t^2 + \ldots + \left(\lambda^{(m-1)j} + \lambda^{-(m-1)j}\right) t^{(m-1)} + 0 \cdot t^m$$
$$- \ldots - \left(\lambda^{(m-1)j} + \lambda^{-(m-1)j}\right) t^{(m+1)} - \left(\lambda^{2j} + \lambda^{-2j}\right) t^{2m-2} - \left(\lambda^j + \lambda^{-j}\right) t^{2m-1} - t^{2m}.$$
Now, since $\lambda^{(m-n)j} = \lambda^{-nj}$ *for any integer n, we can rewrite this as*
$$\left(1-t^2\right) p\left(\lambda^j t\right) p\left(\lambda^{-j} t\right) = \left(1-t^m\right)\left(1 + \sum_{n=1}^{m-1}\left(\lambda^{jn} + \lambda^{-jn}\right) t^n + t^m\right) = \left(1-t^m\right)\left(\sum_{n=-(m-1)}^{m-1} \lambda^{jn} t^{|n|}\right).$$
Q.E.D.

We now complete the proof of Theorem 2.8
Let $q\left(t_1,\ldots,t_k\right) := \frac{1}{m} \sum_{j=0}^{m-1} \prod_{s=1}^{k} \sum_{i=-(m-1)}^{m} \lambda^{ij} t_s^{|i|}$.
Then $q\left(t_1,\ldots,t_k\right) := \frac{1}{m} \sum_{j=0}^{m-1} \sum_{\left(i_1,\ldots,i_k\right)}^{m} \left(\lambda^j\right)^{\sum_{s=1}^{k} i_s} \prod_{s=1}^{k} t_s^{|i_s|}$, *where* $\left(i_1,\ldots,i_k\right)$ *run through all choices of* $-(m-1) \le i_s \le m$ *for all* $1 \le s \le k$.
Since $\frac{1}{m} \sum_{j=0}^{m-1} \left(\lambda^j\right)^i = \begin{cases} 1, & if\ i \equiv 0 \pmod m \\ 0, & otherwise \end{cases}$ *it follows that* $q\left(t_1,\ldots,t_k\right) := \sum_{\left(i_1,\ldots,i_k\right)\in H}^{m} \prod_{s=1}^{k} t_s^{|i_s|}$,
where H is as in the statement of the Theorem 2.8.
Thus,
$$F_D\left(\bar{t}\right) = \frac{m}{2m} \prod_{s=1}^{k} \frac{1}{\left(1-t_s^m\right)\left(1-t_s^2\right)} \left[q\left(t_1,\ldots,t_k\right) + \prod_{s=1}^{k}\left(1-t_s^m\right)\right]$$
and

$$\frac{F_D(\bar{t})}{F_T(\bar{t})} = \frac{1}{2}\left[\sum_{\bar{i}\in H} t_1^{|i_1|}\cdots t_k^{|i_k|} + \prod_{s=1}^{k}\left(1-t_s^m\right)\right].$$

Q.E.D

The following theorem gives the connection between $F_D(\bar{t}) / F_T(\bar{t})$ *and the flag major index for the general k.*

Theorem 2.10. For $G = I_2(m)$ *acting on* $P_2^{\otimes k}$, $(k, m \geq 1)$,

$$\frac{F_D(\bar{t})}{F_T(\bar{t})} = \sum_{\substack{g_1\cdots g_k = Id \\ g_i \in I_2(m)}} t_1^{fmaj(g_1)}\cdots t_k^{fmaj(g_k)}$$ *where fmaj is as defined in (9).*

Proof: *Let H be as in Theorem 2.8. Define the following subsets of H:*

$$H_1 := \left\{(i_1,\ldots,i_k)\in H \;\middle|\; \exists 1\leq s\leq k,\; i_s\notin\{0,m\} \;\; and \;\; sign\left(i_{s_1}\right) = \varsigma_1\right\}$$

where $s_1 = \min\{s \mid i_s \notin \{0,m\}\}$ is =2 *and* $\varsigma_1 = (-1)^{\#\{t<s_1 \mid i_t = m\}}$.

$$H_2 := \left\{(i_1,\ldots,i_k)\in H \;\middle|\; (\forall s)\;\; i_s\in\{0,m\},\;\; \#\{s \mid i_s = m\}\equiv 0 \pmod 2\right\}.$$

The proof will proceed by the following Lemmas 2.11 - 2.15.

Lemma 2.11.

$$\frac{1}{2}\left(\sum_{\bar{i}\in H} t_1^{|i_1|}\cdots t_k^{|i_k|} + \prod_{s=1}^{k}\left(1-t_s^m\right)\right) = \sum_{\bar{i}\in H_1\cup H_2} t_1^{|i_1|}\cdots t_k^{|i_k|}.$$

Proof:

H can be divided into two parts:

part1 : not all components of $\bar{i}$ *belong to* $\{0,m\}$.

part2 : all components of $\bar{i}$ *belong to* $\{0,m\}$.

Clearly $\prod_{s=1}^{k}\left(1-t_s^m\right)\varsigma_1 = \sum_{\bar{i}\in part2} (-1)^{\#\{s \mid i_s = m\}} t_1^{i_1}\cdots t_k^{i_k}$, *and therefore*

$$\sum_{\bar{i}\in part2} t_1^{|i_1|}\cdots t_k^{|i_k|} + \prod_{s=1}^{k}\left(1-t_s^m\right) = 2\sum_{\bar{i}\in H_2} t_1^{|i_1|}\cdots t_k^{|i_k|}.$$

Also $H_1 \subseteq part1$ *and* $\sum_{\bar{i}\in part1} t_1^{|i_1|}\cdots t_k^{|i_k|} = 2\sum_{\bar{i}\in H_1} t_1^{|i_1|}\cdots t_k^{|i_k|}$

since the map $\varphi : H_1 \to part1 \setminus H_1$ *which reverses the signs of all* $i_s \notin \{0,m\}$ *is clearly a bijection preserving* $t_1^{|i_1|}\cdots t_k^{|i_k|}$.

Q.E.D.

As we already explained in section 3.1, the elements of $I_2(m)^k$ *are*

$$(g_1,\dots,g_k)=\left(r_1^{\alpha_1}f_1^{\beta_1},r_2^{\alpha_2}f_2^{\beta_2},\dots,r_k^{\alpha_k}f_k^{\beta_k}\right),$$

where $r_s f_s \in G_s \cong I_2(m)$ *are fixed, and* $0\le\alpha_s<m,\quad 0\le\beta_s<2,\quad (1\le s\le k)$.

The condition $g_1\cdots g_k=e$ *translates, using the product rules (11), to*

$$\begin{cases}\alpha_1+(-1)^{\beta_1}\alpha_2+(-1)^{\beta_1+\beta_2}\alpha_3+\ldots+(-1)^{\beta_1+\ldots+\beta_{k-1}}\alpha_k\equiv 0\pmod m\\ \beta_1+\ldots+\beta_k\equiv 0\pmod 2\end{cases}$$

Define now

$$S:=\left\{\left((\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_k,\beta_k)\right)\middle|\begin{array}{l}0\le\alpha_s<m,\quad 0\le\beta_s<2\ (\forall s)\\ \sum_{s=1}^{k}\beta_s\equiv 0\pmod 2\\ \sum_{s=1}^{k}(-1)^{\beta_1+\ldots+\beta_{s-1}}\alpha_s\equiv 0\pmod m\end{array}\right\}.$$

Clearly, $|S|=(2m)^{k-1}$.

Lemma 2.12. *If* $0\le\beta_s<2\quad(\forall s)$ *and* $\sum_{s=1}^{k}\beta_s\equiv 0\pmod 2$, *then* $\sum_{s=1}^{k}(-1)^{\beta_1+\ldots+\beta_{s-1}}\beta_s\equiv 0$.

Proof: *Consider only the values of* s *with* $\beta_s=1$. *By assumption, there is an even number of them:* $\beta_{s_1}=\ldots=\beta_{s_{2d}}=1$, *and all the others are zero. Then* $\sum_{s=1}^{k}(-1)^{\beta_1+\ldots+\beta_{s-1}}\beta_s=(-1)^0\beta_{s_1}+(-1)^1\beta_{s_2}+\ldots+(-1)^{2d-1}\beta_{s_{2d}}=1-1+\ldots+1-1=0$.

Q.E.D.

Corollary 2.13.

$$S:=\left\{\left((\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_k,\beta_k)\right)\middle|\begin{array}{l}0\le\alpha_s<m,\quad 0\le\beta_s<2\ (\forall s)\\ \sum_{s=1}^{k}\beta_s\equiv 0\pmod 2\\ \sum_{s=1}^{k}(-1)^{\beta_1+\ldots+\beta_{s-1}}\alpha_s\equiv 0\pmod m\end{array}\right\}$$

Let

$$S_1 := \left\{ \left((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k) \right) \in S \,\middle|\, \exists 1 \le s \le k, \quad (\alpha_s,\beta_s) \notin \{(0,0),(m-1,1)\} (\forall s) \right\}$$

$$S_1 := \left\{ \left((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k) \right) \in S \,\middle|\, \quad (\alpha_s,\beta_s) \in \{(0,0),(m-1,1)\} (\forall s) \right\}.$$

Clearly, $S = S_1 \cup S_2$ *(disjoint union).*

Now we want to prove that $\frac{F_D(\bar{t})}{F_T(\bar{t})} = \sum_{\bar{i} \in H_1 \cup H_2} t_1^{|i_1|} \cdots t_k^{|i_k|} = \sum_{((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k)) \in S_1 \cup S_2} t_1^{\alpha_1+\beta_1} \cdots t_k^{\alpha_k+\beta_k}$.

Define a map $\phi : S_1 \cup S_2 \to H_1 \cup H_2$ *by*

$\phi\left((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k) \right) := (i_1,\dots i_k)$, *where*

$$i'_s := (-1)^{\beta_1+...+\beta_{s-1}} (\alpha_s+\beta_s) \quad (1 \le s \le k) \text{ and } i_s := \begin{cases} -i'_s, & \text{if } i'_s = m; \\ i'_s, & \text{otherwise.} \end{cases}$$

Lemma 2.14. *The map _ is surjective.*

Proof: *Clearly* $\phi(S_2)$ *is equal to* H_2. *It is suficient to show that for any* $(i_1,\dots i_k) \in H_1$ *there exists an element* $(\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k) \in S_1$ *such that*

$$\phi\left((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k) \right) := (i_1,\dots i_k).$$

We determine (α_s,β_s) *by recursion on s.*

Let $1 \le s \le k$ *and suppose* (α_t,β_t) *have been determined for all* $t < s$.

(1) *If* $i_s \in \{0,m\}$ *then* (α_s,β_s) *is determined uniquely:*

If $i_s = 0$, *let* $(\alpha_s,\beta_s) = (0,0)$;

If $i_s = m$, *let* $(\alpha_s,\beta_s) = (m-1,1)$.

(2) *If* $i_s \notin \{0,m\}$ *then there are two possibilities:*

either $(\alpha_s,\beta_s) = (|i_s|,0)$ *or* $(\alpha_s,\beta_s) = (|i_s|-1,1)$.

The decision between these two options depends on the sign of $i_{\bar{s}}$, *where* $\bar{s} := \min\{t \mid t > s, i_t \notin \{0,m\}\}$ *(see below the case when* $\bar{s}$ *is undefined).*

For any $s < t < \bar{s}$, $i_s \in \{0,m\}$. *Thus* $\beta_s \in \{0,1\}$ *is determined so as to satisfy*

$$sign(i_{\bar{s}}) = sign(i_s)(-1)^{\beta_s + \#\{s<t<\bar{s} : i_t = m\}}.$$

In other words: if $sign(i_{\bar{s}}) = sign(i_s)(-1)^{\#\{s<t<\bar{s}: i_t = m\}}$, *let* $(\alpha_s, \beta_s) := (|i_s|, 0)$; *otherwise* $(\alpha_s, \beta_s) := (|i_s| - 1, 1)$.

If $\bar{s}$ *is undefined (this happens for exactly one* s *, the last one with* $i_s \notin \{0, m\}$ *),*

then the decision depends on the condition $\sum_{s=1}^{k} \beta_s \equiv 0 \pmod 2$,

namely: if $sign(i_s)(-1)^{\#\{s<t<\bar{s}: i_t = m\}} = 1$ *let* $(\alpha_s, \beta_s) := (|i_s|, 0)$; *otherwise* $(\alpha_s, \beta_s) := (|i_s| - 1, 1)$.

Clearly, this recursive definition gives $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_k, \beta_k) \in S$ *with* $\phi((\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_k, \beta_k)) := (i_1, \dots i_k)$.

Q.E.D.

Example*: Let* $m = 5, k = 8$ *and* $(2, -3, 0, 5, -1, -4, 3, 3) \in H_1$.

We need to find $(\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_8, \beta_8) \in S$ *such that*

$\phi((\alpha_1, \beta_1), (\alpha_2, \beta_2), ..., (\alpha_8, \beta_8)) := (2, -3, 0, 5, -1, -4, 3, 3)$.

1. To determine (α_1, β_1) *(either* $(2,0)$ *or* $(1,1)$*), we see that for* $s = 1, \bar{s} = 2$: $sign(i_1) \neq sign(i_2)$. *Thus* $(\alpha_1, \beta_1) = (1,1)$.

2. To determine (α_2, β_2) *(either* $(3,0)$ *or* $(2,1)$*), we see that for* $s = 2, \bar{s} = 5$:

$sign(i_5) \neq sign(i_2)$. *Thus* $(\alpha_2, \beta_2) = (2,1)$.

3. Since $i_3 = 0$, $(\alpha_3, \beta_3) = (0,0)$.

4. Since $i_4 = 5$, $(\alpha_4, \beta_4) = (4,1)$

5. To determine (α_5, β_5) *(either* $(1,0)$ *or* $(0,1)$*), we see that for* $s = 5, \bar{s} = 6$: $sign(i_5) = sign(i_6)$. *Thus* $(\alpha_5, \beta_5) = (1,0)$.

6. To determine (α_6, β_6) *(either* $(4,0)$ *or* $(3,1)$*), we see that for* $s = 6, \bar{s} = 7$: $sign(i_7) \neq sign(i_6)$. *Thus* $(\alpha_6, \beta_6) = (3,1)$.

7. $i_7 = 3 > 0$ *and* $i_8 = 3 > 0$; *the sign has not changed. Thus*

$(\alpha_7, \beta_7) = (3,0)$.

8. The last element $i_8 = 3$ *determines* (α_8, β_8).

Since $\sum_{s=1}^{8} \beta_s = 1+1+0+1+0+1+0+\beta_8 \equiv 0 (\bmod 2)$, we get $\beta_8 = 0$ and $(\alpha_8, \beta_8) = (3,0)$.

It is easy to verify that indeed:

$$\phi((1,1),(2,1),(0,0),(4,1),(1,0),(3,1),(3,0),(3,0)) = (2,-3,0,5,-1,-4,3,3)$$

.

Since clearly $|S_1 \cup S_2| = |H_1 \cup H_2| = (2m)^{k-1}$; we conclude

Corollary 2.15. ϕ is a bijection, and if $\phi((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k)) := (i_1,\dots i_k)$, *then*

$$t_1^{|i_1|} \cdots t_k^{|i_k|} = t_1^{\alpha_1+\beta_1} \cdots t_k^{\alpha_k+\beta_k}.$$

End of Proof of Theorem 2.10:

Combining Theorem 2.8, Lemma 2.11 and Corollary 2.15 gives

$$\frac{F_D(\bar{t})}{F_T(\bar{t})} = \sum_{((\alpha_1,\beta_1),(\alpha_2,\beta_2),...,(\alpha_k,\beta_k)) \in S} t_1^{\alpha_1+\beta_1} \cdots t_k^{\alpha_k+\beta_k}$$

Recall: $\alpha_i + \beta_i = fmaj(g_i), 1 \le i \le k$ *by (9) the conclusion of Corollary 2.15 is exactly the statement of the Theorem 2.10.*

Q.E.D.

3. Explicit Bases

3.1. Diagonal invariants for *k* = 2. The rank of the diagonal invariant algebra (*DIA*) as a free module over the tensor invariant algebra (*TIA*) is *2m*, [12]. We are looking for a corresponding "basis" of diagonal invariants.

Claim 3.1. *Let* $G = I_2(m)$ *act naturally on* $P_n^{\otimes 2}$. *Then TIA, the tensor invariant algebra, is a polynomial ring in the 4 polynomials*

$$E_T = \{\theta_1 = x_1 y_1, \theta_2 = x_1^m + y_1^m, \theta_3 = x_2 y_2, \theta_4 = x_2^m + y_2^m\} \subseteq R.$$

Theorem 3.2. *Let* $G = I_2(m)$ *act naturally on* $P_n^{\otimes 2}$.

Then $B = \{1,\ x_1^i x_2^{m-i} + y_1^i y_2^{m-i},\ x_1^j y_2^j + y_1^j x_2^j \ |1 \le i \le m-1, 1 \le j \le m\}$ *is a basis for the diagonal invariant algebra (DIA) as a free module over the tensor invariant algebra (TIA).*

Proof: *Let R = DIA be the ring of polynomials invariant under the diagonal action of* G *on* $P_n \otimes P_n$ *. It is clear that the following set is an in_nite basis of* R *as a vector space over the ambient field* F: $\overline{B} := \left\{ x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} + x_1^{j_1} y_1^{i_1} x_2^{j_2} y_2^{i_2} \;\middle|\; i_1, i_2, j_1, j_2 \geq 0, \quad i_1 + i_2 \equiv j_1 + j_2 \left(\mathrm{mod}\, m\right) \right\}$.

$\overline{B}$ *is obtained by applying the Rayleigh operator* $\frac{1}{|G|}\sum_{g\in G}\rho(g)$ *to all monomials in* $P_n^{\otimes 2}$ *omitting repetitions and zero polynomials.*

Claim 1: *B spans DIA as a module over TIA.*

This means that every polynomial $p \in DIA$ *can be represented as a linear combination of elements of B, with coefficients which are polynomials in the elements of* E_T

(16) $$p = \sum_{i=1}^{2m} b_i P_i \left(\theta_1, \theta_2, \theta_3, \theta_4\right), \text{ where } b_i \in B \quad \left(\forall i\right).$$

Since R is spanned by $\overline{B}$ *over a field, it is enough to show that (16) holds for any element* $p \in \overline{B}$. *We shall prove (16) by induction on deg(p).*

If deg(p) = 0, then p is a constant $p = 1 \cdot c$ *and the statement is true (*$1 \in B$*).*

Assume that (16) holds for deg(p) < N. We shall prove that it holds also for deg(p) = N > 0.

Case 1: *Either* $i_1 j_1 \neq 0 \quad or \quad i_2 j_2 \neq 0$.

If, for example, $i_1 j_1 \neq 0$ $\quad p = x_1^{i_1} y_1^{j_1} x_2^{i_2} y_2^{j_2} + x_1^{j_1} y_1^{i_1} x_2^{j_2} y_2^{i_2} = \theta_1 p_1$, *where* $\theta_1 = x_1 y_1$, *and* $p_1 = x_1^{i_1-1} y_1^{j_1-1} x_2^{i_2} y_2^{j_2} + x_1^{j_1-1} y_1^{i_1-1} x_2^{j_2} y_2^{i_2}$. *Clearly* $p_1 \in B$ *and deg(*p_1*) < deg(p) = N. Thus (16) holds. In all remaining cases, at most two of* i_1, i_2, j_1, j_2 *are nonzero.*

Case 2: $i_1 = i_2 = 0 \quad and \quad j_1 j_2 \neq 0 \quad \left(or \quad j_1 = j_2 = 0 \quad and \quad i_1 i_2 \neq 0\right)$.

Then $\quad p = y_1^{j_1} y_2^{j_2} + x_1^{j_1} x_2^{j_2}, \quad \left(j_1, j_2 > 0,\; j_1 + j_2 \equiv 0 \left(\mathrm{mod}\, m\right)\right)$.

If $j_1 + j_2 = m$, *then* $p = y_1^{j_1} y_2^{m-j_1} + x_1^{j_1} x_2^{m-j_1} \in B$.

If $j_1, j_2 > m$, *then If* $j_1 + j_2 \geq 2m$. *Assume, with no loss of generality, that* $j_1 \geq j_2$, *then* $j_1 \geq m$.

If $j_1 > m$ *then*

$$p = \left(y_1^m + x_1^m\right)\left(y_1^{j_1-m} y_2^{j_2} + x_1^{j_1-m} x_2^{j_2}\right) - x_1^m y_1^{j_1-m} y_2^{j_2} - y_1^m x_1^{j_1-m} x_2^{j_2} = \theta_2 q_1 + \theta_1 q_2,$$

where $\theta_1 = x_1 y_1, \theta_2 = x_1^m + y_1^m \in E_T$ *and* $q_1, q_2 \in R$ satisfy $\deg\left(q_1\right), \deg\left(q_2\right) < N$.

Thus (16) holds.

On the other hand, if $j_1 = j_2 > m$ *then* $p = \left(y_1^m + x_1^m\right)\left(y_2^m + x_2^m\right) - x_1^m y_2^m - y_1^m x_2^m = \theta_2\theta_4 - q_2,$

where $\theta_2 = x_1^m + y_1^m, \theta_4 = x_2^m + y_2^m, \in E_T$ *and* $q_2 = x_1^m y_2^m - y_1^m x_2^m \in R$. *Thus (16) holds again.*

Case 3: *Either* $i_1 = j_2 = 0$ *and* $i_2 j_1 \neq 0$ $($*or* $i_2 = j_1 = 0$ *and* $i_1 j_2 \neq 0$ $)$

Then $p = x_2^{i_2} y_1^{j_1} + y_2^{i_2} x_1^{j_1}$, $\left(i_2, j_1 > 0,\ i_2 \equiv j_1 \left(\bmod m\right)\right)$.

Otherwise, we may assume that $i_2 \le j_1$ *and* $j_1 > m$, then $p = x_2^{i_2} y_1^{j_1} + y_2^{i_2} x_1^{j_1} = \left(x_1^m + y_1^m\right)\left(x_2^{i_2} y_1^{j_1-m} + y_2^{i_2} x_1^{j_1-m}\right) - x_1^m x_2^{i_2} y_1^{j_1-m} - y_1^m y_2^{i_2} x_1^{j_1-m} = \theta_2 q_1 - \theta_1 q_2,$

where $\theta_1, \theta_2 \in E_T$ *and* $\deg\left(q_1\right), \deg\left(q_2\right) < N$. *Thus (16) holds.*

Case 4: Exactly one of i_1, i_2, j_1, j_2 is nonzero.

With no loss of generality, $p = x_1^{i_1} + y_1^{i_1}$, $\left(i_1 > 0,\ i_1 \equiv 0\left(\bmod m\right)\right)$.

If $i_1 = m$ *then* $p = x_1^m + y_1^m = \theta_2$.

If $i_1 > m$ *then* $p = \left(x_1^m + y_1^m\right)\left(x_1^{i_1-m} + y_1^{i_1-m}\right) - x_1^{i_1-m} y_1^m - y_1^{i_1-m} x_1^m = \theta_2 q_1 + \theta_1 q_2,$

Where again $\theta_1, \theta_2 \in E_T$, $q_1, q_2 \in R$ *and* $\deg\left(q_1\right), \deg\left(q_2\right) < N$. *Thus (16) holds.*

This concludes the proof of Claim 1.

Claim 2*: B is a basis for DIA as a free module over TIA. Every element of B is the set of invariants under the diagonal action, and* $|B| = 2m$. *For any* $b_i \in B$, *the graded vector space* $V_i := b_i \cdot TIA \subseteq R$ *has Hilbert series* $h_i\left(\bar{t}\right) F_T\left(\bar{t}\right)$, *where* $h_i\left(\bar{t}\right)$ *is the Hilbert monomial corresponding to* b_i.

Since $R = DIA = V_1 + \ldots + V_{2m}$ *by Claim 1,* $F_R\left(\bar{t}\right) \le \sum_{i=1}^{2m} h_i\left(\bar{t}\right) F_T\left(\bar{t}\right)$ *with equali if and only if the sum* $V_1 + \ldots + V_{2m}$ *is direct, i.e., B is a basis for DIA as a module over TIA.*

A direct computation shows that indeed $\sum_{i=1}^{2m} h_i\left(\bar{t}\right) = \sum_{i=0}^{m} t_1^i t_2^i + \sum_{i=1}^{m-1} t_1^i t_2^{m-i} = \dfrac{F_D\left(\bar{t}\right)}{F_T\left(\bar{t}\right)}$

(the latter equality-by Theorem 2.8), and this complete the proof of Theorem 3.2.

Q.E.D.

3.2. A basis for the coinvariant algebra. *Let* $G = I_2(m)$ *act naturally on* $P_2 = S(V)$.

By Theorem 2.4 the ring of invariants $S(V)_G$ *is freely generated by* $\theta_1 = x_1 y_1, \theta_2 = x_1^m + y_1^m$. *Let* $I = \langle \theta_1, \theta_2 \rangle$ *be the ideal of* P_2 *generated by* θ_1, θ_2 *then* P_2/I *is the coinvariant algebra*

of $I_2(m)$*). By [15, 3.15], the Hilbert function of* P_2/I *is*

$$\frac{H(P_2,t)}{H(I,t)} = \frac{(1-t^2)(1-t^m)}{(1-t)^2} = 1 + 2(t + ... + t^{m-1}) + t^m = \sum_{g \in I_2(m)} t^{fmaj(g)}.$$

Lemma 3.3. $B = \{ y^j, x^{j+1} \ | 0 \le j \le m-1 \}$ *is a basis for the coinvariant algebra.*

Proof: *Let* $R = P_2/I$. *It is clear that the following set is an infinite basis of R as a vector space over the ambient field* F *:* $B := \{ x^i y^j + I, \ | i, j \ge 0 \}$.

Claim 1: *B spans* P_2/I

This means that every polynomial $p \in P_2/I$ *can be represented as a linear combination of elements of B, with coefficients which are polynomials in the elements of* I.

(17) $p = \sum_{i=1}^{2m} b_i P_i \ (\theta_1, \theta_2)$, *where* $b_i \in B \quad (\forall i)$.

Since R *is spanned by* $\overline{B}$ *over a field, it is enough to show that (17) holds for any element* $p \in \overline{B}$. *We shall prove (17) by induction on* $\deg(p)$.

If $\deg(p) = 0$, *then* p *is a constant* $p = 1 \cdot c$ *and the statement is true (*$1 \in B$*).*

Assume that (17) holds for $\deg(p) < N$. *We shall prove that it holds also for deg(p) = N > 0.*

Case 1: Exactly one of i, j *is nonzero, with no loss of generality, then* $p = y^j \quad (j > 0)$.

If $j = m$ *, then* $p = y^m = \theta_2 - x^m$, *where* $\theta_2 = x^m + y^m \in I$ *and* $x^m \in B$ *.*

If $j > m$ *, then* $p = (y^m + x^m)(y^{j-m}) - x^m y^{j-m} = \theta_2 q_1 + \theta_1 q_2$, *where* $\theta_1 = xy \in I$, $\theta_2 = x^m + y^m \in I$, $q_1, q_2 \in R$ *and* $\deg(q_1), \deg(q_2) < N$. *Thus (17) holds.*

If $j < m$ *, then* $p = y^j \in B$, *thus (17) holds.*

Case 2: $i \neq 0$ *and* $j \neq 0$, *then* $p = x^i y^j \quad (i, j > 0)$

If $0 < i = j \leq m$, then $p = x^i y^i \in I$.

Otherwise, we may assume that $i \leq j$ *and* $j > m$, *then* $p = x^i y^j = (x^m + y^m)(x^i y^{j-m}) - x^{m+i} y^{j-m} = \theta_2 q_1 + \theta_1 q_2$,

here $\theta_1, \theta_2 \in I$, $q_1, q_2 \in R$ *and* $\deg(q_1), \deg(q_2) < N$. *Thus (17) holds.*

If $i \leq j$ *and* $j < m$, *then* $p = x^i y^j = x^i y^i (y^{j-i}) = \theta_1 q$, *where* $\theta_1 \in I$, $q \in R$ *and* $\deg(q) < N$. *Thus (17) holds.*

This concludes the proof of Claim 1.

Claim 2: B *spans* P_2/I. *Since* $|B| = |I_2(m)| = 2m$. *For any* $b_i \in B$, *the graded vector space* $V_i := b_i \cdot I \subseteq R$ *has Hilbert series* $h_i(\bar{t})I$, *where* $h_i(\bar{t})$ *is the Hilbert monomial corresponding to* b_i. *Since* $R = P_2/I = V_1 + ... + V_{2m}$ *by Claim 1,* $H(R, \bar{t}) \leq \sum_{i=1}^{2m} h_i(\bar{t}) I$ *with equality if and only if the sum* $V_1 + ... + V_{2m}$ *is direct, i.e.,* B *is a basis for* P_2 *as a module over* I. *This complete the proof of Lemma 3.3.*

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