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THE STRUCTURE OF A PI-RINGS

О. Овруцкая. В этой статье мы покажем индуктивный процесс, который дает нам возможность доказать основные результаты в структуре колец. Некоторые из этих результатов будут новыми, остальные короткими доказательствами известных результатов, таких как теоремы Капланского. (Эта статья является частью кандидатской диссертации на степень магистра в университете Бар-Илан в Рамат-Гане, IL)

***O. Ovrutzki** , I*n this paper we show an inductive process, which enable us to prove basic results in the structure of rings. Some of these results will be new, and others will be short direct proofs of well-known results like Kaplansky's theorem. This article is a part of the candidate dissertation for M.SC. in Bar Ilan University, Ramat Gan, IL)

Introduction: The main method we use in this paper is to pass to *W=Ra/Ann a*, where *a* is a suitably chosen element of *R*. To this end, we must compare the structure of *R* and *W*, in particular the prime and primitive ideals, and also see which degrees and dimensions shrink when passing from R to W, and which do not.We can show W=$Ra/ann(a)$ satisfies polynomial identity of lower total degree. This will enable us to find a short elementary proof of Kaplansky's theorem: Let R be a primitive ring that satisfies a polynomial identity of degree d. Then R has some PI-class $n \le [d/2]$, and $R \approx M_t(D)$ for some division ring D with $n^2 = t^2[D:Z(D)] = [R:Z(R)]$. (See [6], p.36) That is the backbone of PI-theory: any primitive PI-ring is simple Artinian.

The proof is by taking a simple faithful module Ry. Then the left annihilator L of y satisfies a polynomial identity of lower degree, and we proceed by induction.

To sum up, we can formulate the objectives of this project as follows: to develop algorithms for solving theorems about a ring R by means of left ideals.

Primitive PI-rings: We propose the technique for solving problems for rings with polynomial identities (PI-rings). Let *R* be a PI-ring with a one-sided ideal *L*. Let *L* be a left ideal and denote *A=ann L*.

Definitions and basic properties Let *R* be a ring, polynomial $f(x_1,\dots,x_t)$ is called a classical identity of *R*, if $f(r_1,\dots,r_t)=0$ for all $r_1,\dots,r_t\in R$. A classical identity of *R* is called polynomial identity if one of its coefficients is 1; a ring with a polynomial identity is called PI-ring.

Recall: If R is a ring with a polynomial identity $f(x_1,\dots,x_t)$, then f is an identity for each ideal of *R*. A polynomial f is linear in x_i, if x_i occurs exactly once (of degree 1) in every monomial of f. f is *t*-linear if f is linear in $x_1,\dots,x_t$; f is *t*-alternating if $f\left(x_i\to x_j\right)=0$ for all $1\le i\le j\le t$.

A polynomial $f(x_1,\dots,x_t)$ is multilinear in xi has degree 1 in each monomial of f, for each $1\le i\le t$.

The identities of PI-rings. Kaplansky (1948) proved a theorem, that any primitive PI-algebra is simple artinian. In fact this theorem is the backbone of PI-theory. Our aim is to prove it without using the Jacobson's density theorem.

Proposition 1: Let *R* be a primitive ring with identity *f* of degree *n*, *M=Ry* be a faithful *R*-module. If $L=\ell_R(y)$, the left annihilator of *y*, is a left ideal of *R*, then L satisfies an identity of degree *n*-1.

Proof: *R* is a primitive ring, so there exists a simple faithful *R*-module *M*; while *M=Ry*. Let denote $L=\ell_R(y)$, so $Ly=0$. We want to prove that *L* satisfies an identity of degree *n*-1.

Write $f(x_1,\dots,x_t)=\sum_{\in S_n} \mathrm{x}_1\cdots\mathrm{x}_t$. Let f_1 be the sum of those monomials that finish with x_n and $f_2=f-f_1$; thus we may denote $f_1=gx_n$ and write our polynomial *f* as a sum of two polynomials f_1 and f_2: $f=gx_n+f_2$.

For all $r_i\in R, i=1,\dots,n$ implies $f(\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{n})\mathrm{y}=0$, also $f(\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{n})\mathrm{y}=g(\mathrm{r}_1,\dots,\mathrm{r}_{\mathrm{n-1}})\mathrm{r}_n\mathrm{y}+f_2(\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{n})\mathrm{y}$.

Let us take $\mathrm{r}_1,\dots,\mathrm{r}_{\mathrm{n-1}}\in L$ and some $\mathrm{r}_\mathrm{n}\in R$. We see that $f_2(\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{n})\mathrm{y}=0$ since each monomial in $f_2(\mathrm{r}_1,\dots,\mathrm{r}_\mathrm{n})$ ends with some element of *L*.

For all $\mathrm{r}_1,\dots,\mathrm{r}_{\mathrm{n}-1}\in L$ exists $g(\mathrm{r}_1,\dots,\mathrm{r}_{\mathrm{n}-1})\mathrm{Ry}=0$ and *Ry=M* is a faithful module, thus $g(\mathrm{r}_1,\dots,\mathrm{r}_{\mathrm{n}-1})\in\ell_R(\mathrm{y})=0$. Thus $\deg g=\mathrm{n}-1$ and g is an identity of *L* and of $\overline{L}=L/ann_R L$.

We have lowered degree of the polynomial. Using this structure we may get a chain of ideals satisfying identities of lower degrees. This helps us to prove that *R* is simple artinian. But we need one more fact (Theorem 1) to complete the proof.

Recall: Every simple left artinian ring has a minimal left ideal and thus by Wederburn-Artin is of the form $M_n(D)$ for suitable division ring *D*.

Conversely, $R = M_n(D)$ is a simple artinian since we have the composition series of *R*-modules $R > L_{n-1} > L_{n-2} > \ldots > L_0 = 0$, where $L_u = \sum_{i=1}^{u} Re_{ii}$, so $L_u / L_{u-1} \approx Re_{uu}$ is a simple *R*-module.

Note: If *M*=*Ry* is a simple module and $ry \neq 0$, for each $r \in R$, then *R* is a division ring.

Proof: Suppose *M*=*Ry* is a simple module and $0 \neq a \in R$, also $0 \neq ay \in M$. Then $0 \neq Ray \subseteq Ry$, so $Ray = Ry$, and $y \in Ray$, implying $y = bay$ for some $b \in R$. Then for ring *R* with unit element 1, $(1-ba)y = 0 \Rightarrow (1-ba) = 0$ thus $ba = 1$, so we have found a left inverse for every nonzero element. Therefore *R* is a division ring.

If $1 \notin R$, we need to show that *ba* is a unit element, i.e. $bar = r$ and $rba = r$. Since $bary - ry = 0, \forall y \in R \Rightarrow (bar - r)y = 0 \Rightarrow bar = r$.
The same proof is for $rba = r$.

The following results can be found in [6, p.139]

Theorem1: Suppose *R* is a semiprime ring in which every nonzero left ideal contains a minimal left ideal. If *socR* is a sum of a finite number of minimal left ideals then *R* is semisimple artinian.

Proof: If $socR = \sum_{i=1}^{t} L_i$ for minimal left ideals L_i, then we have a composition series $socR = M_{t-1} > M_{t-2} > \ldots > M_1 = 0$ where each $M_u = \sum_{i=1}^{t} L_i$

So $lsocR \leq t$. We shall use this fact to show $socR = Re$ for some idempotent *e*.

Indeed, put $e_0 = 0$ and use the following inductive procedure:

Given e_u an idempotent take a minimal left ideal $L < R(1-e_u)$ and write $L = Ra_u$ for a suitable idempotent a_u of *R*. Writing $a_u = r_u(1-e_u)$ we see $a_u e_u = 0$ but $(1-e_u)a_u$ is a nonzero element; hence $b_u = (1-e_u)a_u$ is an idempotent of *L* orthogonal to e_u, so $e_u + 1 = b_u + e_u$ is idempotent in $socR$.

The chain $0 < Re_1 < Re_2 < \dots$ in $socR$ must have length $\le t$, so $Re_t = socR$.

If $socR$ is not equal to R then by hypothesis $R(1-e_t)$ contains a minimal left ideal not in the socle, which is absurd. Hence $socR = R$.

Proposition 2: Any primitive PI – ring R is simple artinian.

Proof: We use induction on degree and get a primitive artinian ring, thus $R \approx M_t(D)$.

We assume R is a primitive PI- ring (not necessarily with 1), satisfying identity of degree *n*. Since $L = \ell_R(y)$ is a maximal left ideal of R and by proposition1, $\overline{L} = L/ann_R L$ satisfies an identity of degree $n-1$. By induction on PI- degree, $\overline{L}$ is artinian. By induction, $socL$ is a finite sum.

Since $socR \cap L = socL$, and $socL$ is finite dimensional and primitive, there exists a minimal left ideal of *R*, define *Ry=M*. (Now we have $y \in M \subseteq R$)

R is semiprime, so $M^2 \ne 0$; thus $M \not\subseteq L$ and so $L \cap M \ne M$, but $L \cap M$ is submodule of M, so $L \cap M=\{0\}$. $M \subset L + M \Rightarrow L + M = R \Rightarrow$ L+M=R. Hence $L \oplus M = R = socR$ the finite sum of minimal left ideals thus *R* is semisimple, by Theorem 1.

Remark: If $\overline{L} = L/ann_R L = \oplus L_i/ann_R L$ is simple artinian then LL_i is minimal left ideal of R.

Proof: $L/ann_R L = \oplus L_i/ann_R L$, where $L_i/ann_R L$ is minimal in $L/ann_R L$.

Let $a \in L_i$, we want to prove that La_i is minimal left ideal of *R*.

Suppose $L' \subset La_i$, such that $\overline{L'} \ne 0$ (since $LL' \ne 0$).

$\overline{a_i} \in \overline{L'} = L_i/ann_R L \Rightarrow \overline{a_i} \in L' + annL \Rightarrow La_i \subseteq LL' \subseteq L'$.

Proposition 3: If R is an infinite integral domain and the polynomial $f \in R[x_1,\dots,x\ ]$ satisfies $f(r_1,\dots,r_k) = 0$ for all $r_i \in R$, then $f = 0$. [7, p.9]

Corollary: If *R* is a ring, which satisfies an identity of degree *n* and *F* is any field containing *Cent(R)* of *R*. Then $F \otimes R$ satisfies a polynomial identity of degree $\le n$.

Proposition 4: The matrix ring $M_t(D)$ satisfies no polynomial identity of degree n<2t.

Proof: If it does, then $M_t(D)$ satisfies a multilinear identity of degree *n<2t*. Indeed, by multilinearisation we could assume $f(x_1,\dots,x_t) = \sum_{\in S_n} x_1 \cdots x_t$ with $_1 \ne 0$. Set $x_1 = e_{11}, x_2 = e_{12}, x_3 = e_{22}, ...$ where $\{e_{ij} : i, j = 1,\dots,n\}$ is a set of matrix units. Then $f(e_{11},\ldots,e_{1t}) = e_{11}e_{12}\cdots e_{1t} = e_{1t} \neq 0$ while $\sum_{\in S_n} x_1 \cdots x_t = 0$ all of the other permutations of the x 's. This contradicts the assumption that $f(x_1,\ldots,x_t)$ gives an identity for $M_t(D)$.

Split simple artinian rings Suppose *R* is a simple finite dimensional *F*-algebra. *R* is split if $R \approx M_n(F)$ for suitable $n \in N$. A field extension *K* of *F* is a splitting field for *K*, if $R \otimes_F K$ is split as *K*-algebra.

R is a central simple algebra if *R* is simple, ***Z(R)=F*** and $[R:Z(R)] < \infty$.

[6, p.141]

Proposition 5: Any finite dimensional simple *F*-algebra *R* is split, for *F* an algebraically closed field.

Sketch the proof: *R* is simple artinian, so $R \approx M_n(D)$. For any ***d*** in *D*, we see *F*[***d***] is a finite extension of *F* and is field, since ***d*** is algebraic over *F*; thus $D = F$, so $R \approx M_n(F)$.

The following results can be found in [6, p.143-145]

Remark: If *R* is a prime and finite dimensional over its center, then *R* is simple.

Every subalgebra of a finite dimensional division algebra is division algebra.

Proposition 6: If *R* is central simple , then $\overline{F}$ the algebraic closure of F is splitting field for *R*.

Note: $\dim_F R = t^2 \dim_F D$. R is split $\Leftrightarrow$ $t = n$.

Final form of Kaplansky's theorem

Proposition 7: If R is central simple then $[R:Z(R)] = n^2$ for suitable ***n***, in particular, for ***n*** such that $R \otimes_F \overline{F} \approx M_n(\overline{F})$, where $\overline{F}$ is the algebraic closure of *F*. Writing a central simple algebra *R* in the form $M_n(D)$ for *D* a division ring, we have the following information:

1. D itself is central simple with $[D:Z(R)] = (n/t)^2$, where $[R:Z(R)] = n^2$.
2. $R \approx M_n(F) \otimes_F D$.
3. R^{op} is also central simple and Rop $R^{op} \approx M_n(D^{op}) \approx M_n(F) \otimes_F D^{op}$.

[6, p.143]

Proposition 8: Suppose *R* is a primitive ring, satisfying a polynomial identity of degree *d*. Then *Z(R)* the center of *R* is a field, and *R* has dimensionality $n \le [d/2]$ over its center.

Proof: Any polynomial identity of *R* holds also for the subrings of *R* and its homomorphic images. Suppose *R* contains a division ring *D*.
R, regarded as an algebra over *Z(R)* satisfies the same multilinear identity as $R \otimes_F K$. Let *K* be a splitting field for *R* over *Z(R)*. By [2, §5.13, p.226] $R \otimes_F K = K_n$, where $2n \le d$. Since n^2 is the dimension of K_n over *K*, and also is the dimension of *R* over *Z(R)*. Thus $\left[R : Z(R)\right] = n^2 \le [d/2]^2$.

Proposition 9: Suppose *F* is an algebraically closed field, *D* is a division ring, which has a base over *F* of cardinality $< |F|$. Then *D*=*F*.
(In another words, if *F* is not algebraically closed then *D* is algebraic over *F*).
Proof: If $a \in D$ were not algebraic over *F*, then $\{(a- \ \ )-1: \ \ \in F\}$ is *F*-independent [6, p.156], contrary to hypothesis. Thus *F*[*a*] is a finite field extension, so $a \in F$, i.e. *D*=*F*.

Proof of Kaplansky's theorem: Let *F*=*Z(R)*=*Z(D)* and take an algebraically closed field of cardinality $> [R : F] + 1$.Then $R \otimes_F K = R_1$ and satisfies the identity, so $R_1 \approx M_n(D_1)$ for some *n*, and division ring D_1 with $[D_1 : K] = [R_1 : K] = [R : F] < |K|$. Hence $D_1 = K$, by proposition 9, so $R_1 = M_n(K)$ and $n \le [d/2]$.

But then $n = [R : F] = t^2 [D : F]$, and *R* has PI-degree *n*.

Note: *R* is simple artinian, so has the form $M_t(D)$, where *D* is a division ring. To conclude the proof, we need to show $n^2 = t^2 \left[D : Z(D)\right] = \left[R : Z(R)\right]$.

Proof: Suppose *F* is a maximal subfield of *D*, then $R \otimes_{Z(R)} F$ is a simple ring which also satisfies the PI of degree *d*, and thus has the form $M_t(F)$, for

some *t*. By proposition 4, $t \le [n/2]$, so $\left[R \otimes_{Z(R)} F : F\right] \le [n/2]^2$; thus $\left[R : Z(R)\right] \le [n/2]^2$.

Define $B = \{b_1, ..., b_m\}$ to be some basis of *D* over *F*. $B_1 = \{b_1 e_{ij}, ..., b_m e_{ij} : 1 \le i, j \le t\}$ is some basis of $R \approx M_t(D)$ over *F*

$\Rightarrow [R:F]=t^2m$.

An alternative approach would be to study the structure of $D\otimes K$ more closely, where $K\subset D$.

Proposition 10: If a is an algebraic element of D and $K=F[a]$, then there exist zero-divisors in $D\otimes K$.

Proof: Suppose F is a field, K is an extension of F, $K=F[a]$.

$K\otimes K \approx K\otimes F[\quad]/\langle f\rangle \approx K[\quad]/\langle f\rangle$, a is algebraic element, then f is reducible. Thus $K\otimes K$ is not a field.

Proposition 11: If a is not algebraic, then $(a\otimes 1-1\otimes a)$ is not an invertible element in $D\otimes_F K$, where $F[a]\subset K$.

On the contrary, suppose $(a\otimes 1-1\otimes a)\sum \ldots =1\otimes 1$.

Let take f_i, g elements of $F[\quad]$.

$(a\otimes 1-1\otimes a)\sum d_i\otimes (f_i(a)/g(a))=1\otimes 1$

$(a\otimes 1-1\otimes a)\sum d_i\otimes f_i(a)=1\otimes g(a)$

Build the homomorphism $a\rightarrow\quad$, such that $g(\quad)=0$.

$f_i(\quad)\in F$, so $(a\otimes 1-1\otimes\quad)\sum d_i\otimes f_i(\quad)=0$.

If $(a\otimes 1-\quad\otimes 1)\sum d_i f_i(\quad)\otimes 1=0$, then $\sum ad_i f_i(\quad)\otimes 1\sum d_i f_i(\quad)\quad\otimes 1=0$;

Thus $(a-\quad)\sum d_i f_i(\quad)=0 \Rightarrow \sum d_i f_i(\quad)=0$.

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